
The Meaning of Ramanujan Now and For the Future

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ABSTRACT:

In this paper we pay homage to this towering figure whose mathematical discoveries so affected mathematics throughout the twentieth century and into the twenty first.

1. INTRODUCTION:

We remember Ramanujan, three things come most vividly to mind:-(1) Ramanujan was a truly great mathematician; (2) Ramanujan's life story is inspiring; and (3) Ramanujan's life and work give credible support to our belief in the Universality of truth. We shall examine each of these topics in the next three sections. A careful examination of each topic should, at least, give us some inkling of the meaning of Ramanujan.

2. RAMANUJAN'S EARLY WORK:

First and foremost in any assessment of Ramanujan are his research achievements. These are presented in his Collected Papers [38], his Notebooks [39], and the Lost Notebook [40]. The Collected Papers have recently been reissued by AMS Chelsea [38] with 70 pages of commentary provided by Bruce Berndt. In addition, Berndt [19] has provided a five volume account of the Notebooks with all of Ramanujan's assertions proved in full. An edited version of the Lost Notebook [40] is being prepared by Berndt and me ([11] is the first of four volumes).

If I had to choose one word to characterize the impact of Ramanujan's work, it would be "surprise."

Euler's result is very easy to prove; the Rogers-Ramanujan identities were surprisingly troublesome as Hardy notes [26].

The formulae have a very curious history. They were found first in 1894 by Rogers, a mathematician of great talent but comparatively little reputation, now remembered mainly from Ramanujan's rediscovery of his work. Rogers was a fine analyst, whose gifts were, on a smaller scale, not unlike Ramanujan's; but no one paid much attention to anything he did, and the particular paper in which he proved the formulae was quite neglected.

Ramanujan rediscovered the formulae sometime before 1913. He had then no proof (and knew that he had none), and none of the mathematicians to whom I communicated the formulae could find one. They are therefore stated without proof in the second volume of MacMahon's Combinatory analysis.

The mystery was solved, trebly, in 1917. In that year Ramanujan, looking through old volumes of the Proceedings of the London Mathematical Society came accidentally across Rogers's paper.

I can remember very well his surprise, and the admiration which he expressed for Rogers's work. A correspondence followed in the course of which Rogers was led to a considerable simplification of his original proof. About the same time I. Schur, who was then cut off from England by the war, rediscovered the identities again.

Schur published two proofs, one of which is "combinatorial" and quite unlike any other proof known."

3. SURPRISES IN THE LOST NOTEBOOK:

This section begins an aspect of Ramanujan's work that has only recently been verified [9], [10]. It concerns two identities that are so surprising to an insider that I avoided studying them for twenty five years out of fear. Here are these formidable identities. First from the middle of page 26 in Ramanujan's Lost Notebook [40]. It seems to have taken me forever to recognize that both of these results lie in the realm of entire functions of the variable a . To understand something of the depth of these discoveries it is necessary to provide at least an intuitive introduction to entire functions. First of all, an entire function is an analytic function of z that has no singularities in the finite portion of the z plane. Consequently, it has a power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

that converges for all z .

We now turn to E. T. Copson [24] who succinctly describes the analogy of entire functions and polynomials: The most important property of a polynomial is that it can be expressed uniquely as a product of linear factors of the form

$$Az^a(1-z/z_1)(1-z/z_2)(1-z/z_3) \dots (1-z/z_n),$$

where A is a constant, a is a positive integer or zero, and z_1, z_2, \dots, z_n the points, other than the origin, at which the polynomial vanishes, multiple zeros being repeated in the set according to their order. Conversely, if the zeros are given, the polynomial is determined apart from an arbitrary constant multiplier. Now a polynomial is an integral function [i.e. entire function] of a very simple type, its singularity at infinity being a pole. We naturally ask whether it is possible to exhibit in a similar manner the way in which any integral function depends on its zeros.

"This prologue leads to the central fundamental theorem on entire functions, Hadamard's Factorization Theorem [24]. The full theorem is not necessary for us. In fact, we need only the following special case. Hadamard's Factorization Theorem (weak case). Suppose

$f(z)$ is an entire function with simple zeros at $z_1; z_2; z_3; \dots; f(0) = 1$ and $\sum_{n=1}^{\infty} |z_n|^{-1} < \infty$,

the $f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right)$

We note that each series defines an entire function of the complex variable a . In each identity, Ramanujan is presenting the Hadamard Factorization of the function in question and additionally is specifying explicit formulas, or at least approximations, for each of the zeros. Once one understands that this is what is going on, generally we are still far from

understanding why Ramanujan is able to make these assertions about the zeros of these functions.

The full details are presented in [9] and [10], but the main idea involved relies on polynomial approximations of these functions. With a sense of great exhilaration I managed to prove these results in January of 1985. The proof appeared in 1986 [4].

Many years later, in 2000, Ken Ono rekindled my interest by noting that Don Zagier had proved a very similar result and had applied it to an interesting L-function evaluation. Our subsequent explorations led us to the following very general result [14].

Proposition 1:- Suppose that $\sum_{n=0}^{\infty} \alpha n z^n$

is analytic for $|z| < 1$. If α is a complex number for which

- (i) $\sum_{n=0}^{\infty} (\alpha - \alpha n) < +\infty$,
- (ii) $\lim_{n \rightarrow +\infty} n(\alpha - \alpha n) = 0$, then

$$\lim_{z \rightarrow 1^-} \frac{d}{dz} (1-z)f(z) = \sum_{n=0}^{\infty} (\alpha - \alpha n),$$

From this general result, In the ensuing time, Coogan, Lovejoy and Ono [22], [23], [34] have found many further applications of this result. Recently, P. Freitas and I [13] have extended this result to the following theorem involving higher derivatives:

Proposition 2:- Let $f(z) = \sum_{n=0}^{\infty} \alpha n z^n$

be analytic for $|z| < 1$, and assume that for some positive integer p we have that

$$\sum_{n=0}^{\infty} \left[\prod_{j=1}^p (n+j) \right] (\alpha n + p - \alpha n + p - 1)$$

Converges.

$$\sum_{n=0}^{\infty} \left[\prod_{j=1}^{p-1} (n+j) \right] (\alpha - \alpha n + p - 1)$$

converges, where α is a fixed complex number such that

$$\lim_{n \rightarrow \infty} \left[\prod_{j=1}^p (n+j) \right] (\alpha n + p - \alpha) = 0.$$

Then

$$\frac{1}{p} \lim_{z \rightarrow 1^-} \left\{ \frac{d^p}{dz^p} [(1-z)f(z)] \right\} = \sum_{n=0}^{\infty} \left[\prod_{j=1}^{p-1} (n+j) \right] (\alpha - \alpha n + p - 1)$$

Our choice of topics in this section has been dictated to some extent by the oddness or surprise of Ramanujan's discoveries. However, it should be stressed that there are numerous recent spectacular discoveries by Ono, Bringmann, Mahlburg, Zwegers and others related to Ramanujan's mock theta functions [20], [21], [36].

4. RAMANUJAN AND COMPUTATION:

Anyone who studies Ramanujan's Lost Notebook comes away with some appreciation of the importance of computation in Ramanujan's work. Again, G. H. Hardy provides a relevant comment [38].

“His memory, and his powers of calculation, were very unusual, but they could not reasonably be called abnormal”. If he had to multiply two large numbers, he multiplied them in the ordinary way; he would do it with unusual rapidity and accuracy, but not more rapidly or more accurately than any mathematician who is naturally quick and has the habit of computation. There is a table of partitions at the end of our paper. This was, for the most part, calculated independently by Ramanujan and Major MacMahon; and Major MacMahon was, in general, slightly the quicker and more accurate of the two.”

In a monograph on q-series [5] I was led to speculate about Ramanujan and the age of computer algebra:

“Sometimes when studying his work I have wondered how much Ramanujan could have done if he had had MACSYMA or SCRATCH-PAD or some other symbolic algebra package. More often I get the feeling that he was such a brilliant, clever, and intuitive computer himself that he really did not need them.”

However much fun such speculation may be; it is more important that we examine the ways in which Ramanujan's computations and calculations have guided more recent discoveries. Among the letters from Ramanujan that perhaps influenced Hardy's remark, we might mention the one on Ramanujan's empirical evidence for the Rogers-Ramanujan identities [40]. The study of the calculations led to a truncated version of the Bailey Chain method [7] which in turn led to positivity theorems about the differences of successive Gaussian polynomials [8]. However, there is perhaps a more revealing result in his early notebooks [19].

Theorem 3:- (Extended Engel Expansion Theorem). Every $A \in L$ has a finite or convergent (relative to the above norm) series expansion of the form

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \dots a_n},$$

where $a_n \in C[q^{-1}]$, $a_0 = [A]$, $v(a_n) \leq -n$, and $v(a_{n+1}) \leq v(a_n) - 1$.

The series is unique for A (up to constants in C), and it is finite if and only if $A \in C(q)$. In addition, if

$$a_0 + \sum_{j=1}^n \frac{1}{a_1 a_2 \dots a_j} = \frac{p_n}{q_n},$$

where $q_n = a_1 \dots a_n$

then

$$\left\| A - \frac{p_n}{q_n} \right\| \leq \frac{1}{2^{n+1} \|q_n\|}$$

and

$$v \left(A - \frac{p_n}{q_n} \right) = -v(q_{n+1}) \geq \frac{(n+1)(n+2)}{2}$$

In fact, the an are given by $a_n = \left[\frac{1}{A_n} \right]$

where $A_0 = A$, $a_0 = [A]$, and $A_{n+1} = a_n A_n - 1$.

From this expansion theorem they observed empirically that the generalized Engel expansion for

$$\prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

Is, in fact,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q, q)_n}$$

In subsequent work [15], it was shown that this observation can be rigorously established. In addition, while is false, nonetheless, the generalized Engel expansion of the left side of yields precisely the first four terms of the right side before chaos sets in in subsequent terms.

Thus while we cannot be certain how Ramanujan came upon we can note that our knowledge that something's going on here" provides inspiration for the exploitation of results like the generalized Engel expansion.

As a final comment in this section, I hark back to Hardy's comparison of the computational skills of Ramanujan and P. A. MacMahon. Certainly at the beginning of the twentieth century MacMahon was the one other major researcher applying computational studies to problems in the theory of partitions.

MacMahon was perhaps most proud of his method of Partition Analysis [35]. The method fell into disuse because of the massive calculations necessary in its employment. However, with the advent of computer algebra systems, this method has nourished. The most direct applications of MacMahon's ideas are developed in [16]. Related systems have been developed by Stembridge [50] implementing the ideas of Stanley [49] and DeLoera [25] implementing Barvinok's work [18].

5. THE LIFE OF RAMANUJAN:

Almost anyone interested enough in Ramanujan to be reading these words knows the broad outline of Ramanujan's life. There have been many biographies written. In India, there are books by S. R. Ranganathan [43], Suresh Ram [41], and K. Srinivasa Rao [44] to name only three. There are booklets for students by T. Soundararajan [47] and P. K. Srinivasan [48]. S. Ramaseshan published a magnificent article in Current Science [42]. This is a small sampling of the interest in and affection shown to the memory of Ramanujan by his countrymen. In North America, Robert Kanigel's The Man Who Knew Infinity [30] was widely read and translated into German [31] and Korean [32]. Ramanujan was born to a poor Brahmin family in Erode on December 22, 1887 and grew up in Kumbakonam. He took an early interest in mathematics and always excelled in his study

thereof. In 1904, he entered the Government College at Kumbakonam. He continued to do well at mathematics but neglected other subjects. As a result, he lost his scholarship and left the college. By 1913 he was married and employed in a dead-end job as a clerk at the Madras Port Trust. Nonetheless, his mathematical studies continued unabated. At the suggestion of friends, he communicated some of his work to English mathematicians, among them G. H. Hardy; Hardy very quickly perceived the depth and brilliance of Ramanujan's achievements. Hardy arranged for Ramanujan to come to England in 1914. For the next several years, these two collaborated on truly path breaking research. They found the formula for the partition function, literally founded probabilistic number theory, and made clear the major role that modular forms could play in several aspects of number theory. In 1917, Ramanujan became ill. It was thought at the time that he had tuberculosis. After convalescing in England through much of 1918, he returned to India in 1919 and died in 1920. It was during this final year that the Lost Notebook (actually a collection of approximately 100 sheets of paper) was written. In speaking of that last year, S. R. Ranganathan quotes Janaki Ammal, Ramanujan's widow [43].

"He returned from England only to die, as the saying goes. He lived for less than a year. Throughout this period, I lived with him without break. He was only skin and bones. He often complained of severe pain. In spite of it he was always busy doing his Mathematics. That evidently helped him to forget the pain. I used to gather the sheets of paper which he filled up. I would also give the slate whenever he asked for it. He was uniformly kind to me. In his conversation he was full of wit and humour. Even while mortally ill, he used to crack jokes. One day he confided in me that he might not live beyond thirty five and asked me to meet the event with courage and fortitude. He was well looked after by his friends. He often used to repeat his gratitude to all those who had helped him in his life."

Ramanujan's life is truly inspiring. The rise from poverty to international acclaim is breathtaking. His courage facing death was magnificent. However, his contributions were not just for his time; they affected mathematics and mathematicians for the next century. It is, of course, possible to be so overwhelmed by the Ramanujan story that one loses the feeling that it might possess lessons for one's own life. S. R. Ranganathan [43] tells the following appealing story which is more down to earth: It was February 1912. K S Srinivasan, popularly called "Sandow" by his friends and a classmate of mine in the Madras Christian College, had known Ramanujan intimately while at Kumbakonam. He called on Ramanujan at Summer House one evening.

SANDOW: Ramanju, they all call you a genius.

RAMANUJAN: What! Me a genius! Look at my elbow, it will tell you the story.

SANDOW: What is all this Ramanju? Why is it so rough and black?

RAMANUJAN: My elbow has become rough and black in making a genius of me! Night and day I do my calculation on slate. (It) is too slow to look for a rag to wipe it out with. I wipe out the slate almost every few minutes with my elbow."

6. THE UNIVERSALITY OF TRUTH:

In 1991 in *Minerva*, the sociologist Edward Shils published an article entitled *Reactions on Tradition, Centre and Periphery and the Universal Validity of Science: The Significance of the Life of S. Ramanujan* [46]. Roughly the first half of Shils' article is devoted to an account of various movements (mostly arising in the West) which question the universality of science. Shils' italicized sub-headings are enough to give the flavor:

Ambivalent Marxism: an early forerunner of the denial of the universality of science.
Sociologists' disregard for any questions regarding the validity of scientific knowledge
Meta-history as a denial of the possibility of truthful knowledge
Does every civilization have its own criteria of scientific validity?
Shils (and I) are firmly convinced of the universal validity of science, and Shils turns to Ramanujan to make the case [46]:

“From thinking about Ramanujan, I have concluded that there are no territorial or social or religious or ethnic limitations on the validity of what a scientist discovers. The discoveries of a scientist of one civilization or nationality can be received, assessed and assimilated by scientists of any other civilization or nationality, assuming, of course, that the recipient scientist is sufficiently informed regarding the state of the subject and has the intelligence and scientific training to comprehend what is offered to him. It seems to this outsider that the scientific and mathematical traditions are alive and well in India. Long may they prosper! In reflecting upon and honoring the life and work of Ramanujan, one finds numerous themes to support the nobler ambitions of humanity”.

7. CONCLUSION:

In his Presidential address to the London Mathematical Society in 1936, G. N. Watson spoke movingly of his emotional response to Ramanujan's achievements. I close by quoting his last few paragraphs:

“The study of Ramanujan's work and of the problems to which it gives rise inevitably recalls to mind Lamé's remark that, when reading Hermite's papers on modular functions, “on a la chair de poule.”.

Ramanujan's discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as any of his earlier work, the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance. To his students such discoveries will be a source of delight and wonder until the time shall come when we too shall make our journey to that Garden of Proserpine where

Pale, beyond porch and portal,
Crowned with calm leaves, she stands
Who gathers all things mortal
With cold immortal hands.”

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